Nonperturbative study of generalized ladder graphs in a $\varphi^2\chi$ theory

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The Feynman-Schwinger representation is used to construct scalar-scalar bound states for the set of all ladder and crossed-ladder graphs in a $\varphi^2\chi$ theory in (3+1) dimensions. The results are compared to those of the usual Bethe-Salpeter equation in the ladder approximation and of several quasi-potential equations. Particularly for large couplings, the ladder predictions are seen to underestimate the binding energy significantly as compared to the generalized ladder case, whereas the solutions of the quasi-potential equations provide a better correspondence. Results for the calculated bound state wave functions are also presented.

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One of the important issues in the study of a composite hadronic system at higher energies, is the search for practical and reliable schemes to describe its relativistic dynamics. Our knowledge about the relativistic two-body bound state problem in field theory is almost exclusively based on the application of the ladder approximation to the Bethe-Salpeter equation (BSE) [1,2]. Unfortunately, the general applicability of the ladder theory can be questioned on physical grounds. In particular, the so-called one-body limit does not lead to the Klein-Gordon equation as it ought to. Moreover, gauge invariance can not be satisfied within this approximation. In order to recover these properties, at least the set of all crossed ladder contributions is needed additionally [3–5]. So far, however, the study of the two-body Green function beyond the ladder theory has been considered not feasible in practice. With this situation in mind, several quasi-potential equations (QPEs) have been proposed and studied as possible candidates for an effective theory. Both the ladder BSE as well as various QPEs have been used in numerous studies throughout a wide range of systems, including mesons [6–9], small nuclei [10–12], few-electron atoms [1] and positronium [2].

In constructing the QPEs, one usually chooses the approximations leading to them such that the above mentioned problems are, at least partially, solved. However, due to our ignorance of the behavior of the *full* BSE solutions, it is presently unclear which of the, possibly infinite number of, QPEs provides the best effective description. In this connection it is clearly of interest to have actual solutions available for cases where a larger class of graphs than the ladder series is included in the BSE and that do not suffer from the difficulties inherent to the latter approximation. Such solutions may serve as a testing ground for the various QPE descriptions. Here we present results for the case where in addition also the complete set of all irreducible crossed-ladder graphs is included in the kernel of the BSE, being the minimal set that is free from the above problems. Self-energy and vertex corrections are not taken into account. The inclusion of these contributions are expected not to lead to qualitatively different predictions [13].

In this letter the bound states formed by two scalar particles φ with mass m interacting through the exchange of a third scalar particle χ with mass μ, are determined using the Feynman-Schwinger representation (FSR) [4,13–18]. Starting from the Euclidean action for the above $\varphi^2\chi$ theory

$$S = \int \mathrm{d}^4x \left[(\partial_\lambda \varphi)^2 + m^2\varphi^2 + \tfrac{1}{2}(\partial_\lambda \chi)^2 + \tfrac{1}{2}\mu^2\chi^2 + g\varphi^2\chi \right], \quad (1)$$

we may reconstruct the bound state of two φ-particles with the set of one-meson-exchange and all irreducible crossed-ladder graphs as driving force, by explicitly integrating out the fields in the two-body Green function G. Details of this procedure can be found in Ref. [4]. According to [4], the FSR offers a closed expression for the 'quenched' G (i.e., neglecting the possible occurrence of vacuum fluctuation $\varphi\varphi$-loops) in terms of path integrals over the particle trajectories z and $\bar{z}$ of the two φ particles. Neglecting also the contributions corresponding to the self-energy and vertex corrections, it has the form

$$G = \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}\bar{s} \int (\mathcal{D}z)_{xy} \int (\mathcal{D}\bar{z})_{\bar{x}\bar{y}} \exp\left(-K[z,s] - K[\bar{z},\bar{s}] + V[z,\bar{z},s,\bar{s}]\right), \quad (2)$$

where K and V are given by

$$K[z,s] = m^2 s + \frac{1}{4s}\int_0^1 \mathrm{d}\tau\, \dot{z}_\lambda^2(\tau), \quad (3)$$

$$V[z,\bar{z},s,\bar{s}] = g^2 s\bar{s} \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\bar{\tau}\, \Delta\left(z(\tau) - \bar{z}(\bar{\tau})\right). \quad (4)$$

Our main objective here will be to compare the predictions obtained from Eqs. (2-4) to those from the ladder BSE and various QPEs.

The functional integrations are over all possible paths, subject to the boundary conditions $z(0) = x$, $z(1) = y$ and similarly for $\bar{z}$. In (3+1) dimensions the free two-point function $\Delta(x)$ is given by

$$\Delta(x) = \frac{\mu}{4\pi^2|x|} K_1(\mu|x|). \quad (5)$$

In [4] it was shown that for unequal masses, Eq. (2) satisfies the correct one-body limit. In addition, it was proven that, combined with Eq. (4), it effectively sums up all ladder and, due to the absence of any ordering in the interaction kernel, also all crossed-ladder contributions to G. Each graph of this set is UV-finite, so that no short-distance regularization is required.

The bound state spectrum can be determined by studying the behavior of G with respect to variations of its initial points $(x,\bar{x})$ and final points $(y,\bar{y})$. Considering, in particular, large timelike separations $T = \frac{1}{2}(y_4 + \bar{y}_4 - x_4 - \bar{x}_4)$, we infer from the spectral decomposition

$$G = \sum_{n=0}^{\infty} c_n \exp\left(-m_n T\right) \overset{T\to\infty}{\simeq} c_0 \exp\left(-m_0 T\right), \quad (6)$$

that, asymptotically the Green function is dominated by the ground state contribution.

Notice that the path integrals in Eq. (2) are quantum mechanical ones. This amounts to a considerable reduction in number of degrees of freedom as compared to, for example, putting the field action (1) on a discrete 4-dimensional lattice. As a result accurate calculations can be carried out with this approach also for very large times T.

Let us now briefly discuss the traditional Bethe-Salpeter approach [1,2] to the two-body bound state

problem. In the ladder approximation the wave function Ψ in momentum space obeys the following integral equation

$$S^{-1}(q)\Psi(q) = \frac{\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4q'\, V(q-q')\,\Psi(q'), \qquad (7)$$

where q is the relative momentum between the two φ particles. After a Wick-rotation, the free two-body propagator S and the bare interaction V assume the following form in the CM-frame

$$S(q) = \frac{1}{\left(\mathbf{q}^2+\omega^2+m^2-\frac{1}{4}s\right)^2+s\omega^2}, \qquad (8)$$

$$V(q-q') = g^2\frac{1}{(\mathbf{q}-\mathbf{q}')^2+(\omega-\omega')^2+\mu^2}, \qquad (9)$$

with the relative momentum $q=(\mathbf{q},\omega)$. In the bound state region Eq. (7) only supports solutions for values of the invariant energy $\sqrt{s}$, that correspond to bound states.

Since for unequal masses Eq. (7) in the ladder approximation does not possess the correct one-body limit, several modifications to it have been proposed. Generally, they reduce the description from a 4-dimensional to a 3-dimensional one by making an ansatz for one of the functions involved. This ansatz is chosen such that the resulting quasi-potential equation doès possess the correct one-body limit. Here we study three particular examples: the BSLT-equation [19], the equal-time (ET) equation [8,10,11,20] and the Gross equation [5,12], which have been widely used in the literature.

For the BSLT equation one assumes that the pole structure of the two-body propagator can be approximated via a dispersion relation. Similar to the BSLT case, in the ET prescription the interaction is usually supposed to be independent of the relative time, i.e., also neglecting retardation effects. An additional term is supplied in order to include some of the crossed-box contributions. In doing so, the correct one-body limit is obtained in this approach. Finally, in the Gross formalism one puts one of the two particles on its mass-shell by hand. These procedures lead to the following forms of $S(q)$

$$S_{\mathrm{QPE}}(q) \stackrel{\mathrm{BSLT}}{=} 2\pi\,\delta(\omega)\frac{1}{\sqrt{\mathbf{q}^2+m^2}}\,\frac{1}{\mathbf{q}^2+m^2-\frac{1}{4}s}, \qquad (10)$$

$$\stackrel{\mathrm{ET}}{=} 2\pi\,\delta(\omega)\frac{1}{\sqrt{\mathbf{q}^2+m^2}}\,\frac{1}{\mathbf{q}^2+m^2-\frac{1}{4}s} \times\left(2-\frac{s}{4(\mathbf{q}^2+m^2)}\right), \qquad (11)$$

$$\stackrel{\mathrm{Gross}}{=} 2\pi\,\delta\left(\omega+\tfrac{1}{2}\sqrt{s}-\sqrt{\mathbf{q}^2+m^2}\right) \times\frac{1}{4\sqrt{s}\sqrt{\mathbf{q}^2+m^2}}\,\frac{1}{\sqrt{\mathbf{q}^2+m^2}-\frac{1}{2}\sqrt{s}}. \qquad (12)$$

For all cases the delta-function allows for the elimination of the relative energy variable ω from the description. The ladder BSE and 3-dimensional QPEs were solved by performing a standard partial wave decomposition, thereby factorizing the angular variables.

The FSR solutions were obtained by discretizing the functional integrals, according to

$$(\mathcal{D}z)_{xy} \longrightarrow \left(\frac{N}{4\pi s}\right)^{2N}\prod_{i=1}^{N-1}\int \mathrm{d}^4 z_i. \qquad (13)$$

The normalization in Eq. (13) was chosen such that, when expanded in the coupling g^2, the Green function correctly reproduces the Feynman perturbation series. In terms of the discretized variables the functionals K and V assume the following form

$$K[z,s] \longrightarrow m^2 s + \frac{N}{4s}\sum_{i=1}^{N}(z_i-z_{i-1})^2, \qquad (14)$$

$$V[z,\bar{z},s,\bar{s}] \longrightarrow \frac{g^2 s\bar{s}}{N^2}\sum_{i,j=1}^{N}\Delta\left(\tfrac{1}{2}(z_i+z_{i-1}-\bar{z}_j-\bar{z}_{j-1})\right) \qquad (15)$$

The discretized boundary conditions are $z_0=x$, $z_N=y$ and similarly for $\bar{z}$.

The integral over all degrees of freedom was performed with the Metropolis Monte-Carlo algorithm. The ground state mass can be obtained most efficiently by computing the logarithmic derivative of G instead of G itself

$$L(T) \equiv -\frac{\mathrm{d}}{\mathrm{d}T}\ln\left[G(T)\right] \stackrel{T\to\infty}{\longrightarrow} m_0. \qquad (16)$$

Introducing the shorthand notation Z for the full set of degrees of freedom and putting $S[Z] \equiv K[z,s]+K[\bar{z},\bar{s}]-V[z,\bar{z},s,\bar{s}]$, we may write $L(T)$ as

$$L(T) = \int \mathcal{D}Z\; S'[Z]\,\mathrm{e}^{-S[Z]} \Big/ \int \mathcal{D}Z\,\mathrm{e}^{-S[Z]}, \qquad (17)$$

where the prime denotes an analytical differentiation of the functionals with respect to the endpoint T. According to Eq. (17) the ground state mass is obtained by averaging $S'[Z]$ over an ensemble generated by the action $S[Z]$ for sufficiently large T. The FSR ground state wave function Ψ can readily be found by performing an additional integration of G in Eq. (2) over the spatial relative components $\mathbf{r} \equiv \bar{\mathbf{y}}-\mathbf{y}$ of the final point and incorporating this coordinate in the set Z. By keeping track of the distribution of $|\mathbf{r}|$'s when computing $L(T)$, the $\mathbf{r}$-dependence of Ψ can be determined.

The convergence in N was studied and the mass of the bound state was found to become independent of N at typical values of $N=35$-40. Furthermore, $mT=40$ usually sufficed for $L(T)$ to become independent of T and to reach its asymptotic estimate (16). Since the integrals over s and $\bar{s}$ in Eq. (2) formally diverge for large values, a cutoff $s_{\max}$ had to be introduced in order to render the

functional integrals finite. No dependence on the value of $s_{\max}$ was observed.

In Fig. 1 we present calculations of the ground state mass as a function of the conventional (dimensionless) coupling constant $g^2/4\pi m^2$ for the case $\mu/m = 0.15$.

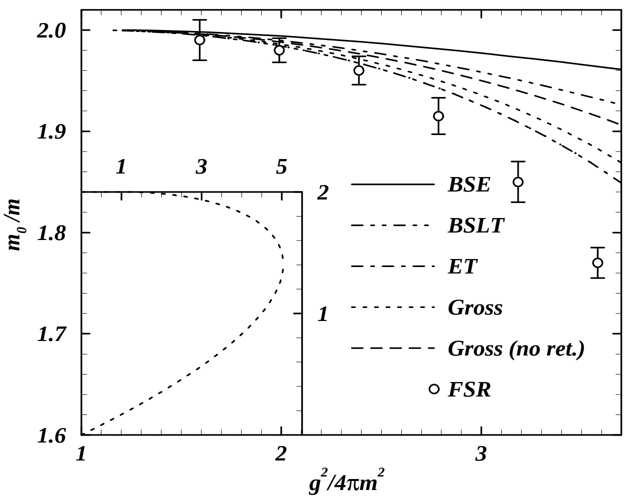

FIG. 1. Ground state mass m_0 of the $\varphi^2\chi$-theory as a function of the dimensionless coupling constant $g^2/4\pi m^2$ for $\mu/m = 0.15$. The inset shows the evolution of the Gross ground state and its unphysical branch over a larger range of couplings.

Since the self-energy contributions have been neglected in the FSR calculations, we may directly compare the predictions to those of the ladder BSE and the various QPEs. The range of validity of the ladder theory is seen to be restricted to the region of small couplings. Generally speaking, for stronger couplings all approximations tend to underbind the system as compared to the FSR results. All QPEs generate more binding energy than the ladder BSE and their results are generally closer to the FSR ones. For the Gross equation we also performed a calculation where the retardation in the interaction was neglected, i.e., we simply put $\omega = \omega' = 0$ in the potential (9). From Fig. 1 we see that in this case the retardation leads to additional attraction. Particularly the ET approximation is seen to give results that relatively provide the best correspondence with the FSR ones.

We remark that due to the energy dependence in the two-body propagator, the Gross equation allows for a second, unphysical solution that starts at $\sqrt{s} = 0$ for $g^2 = 0$ and for which $\sqrt{s}$ grows with increasing g^2. This feature is an artefact of this particular approximation and has also been observed in other but similar dynamical equations [6,21]. Inclusion of negative energy propagation effects was seen to cure this pathological effect. Both the physical and the unphysical solutions are shown in the inset of Fig. 1 and it is seen that they 'annihilate' each other at $g^2/4\pi m^2 \simeq 5.1$, for which $\sqrt{s} \simeq 1.4m$.

In order to compare the FSR ground state wave function to those of the ladder BSE and the various QPEs, we adjust the coupling constants such that the same value of the ground state mass is found. In Fig. 2 we show the ladder BSE and FSR wave functions for relative time $t = 0$ and compare them to the QPE wave functions.

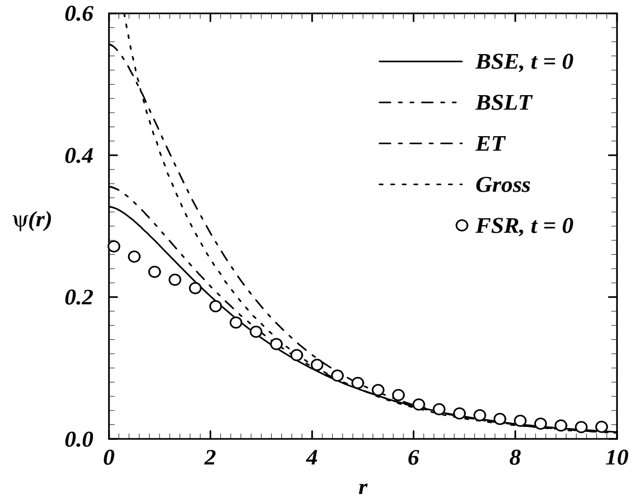

FIG. 2. Equal-time FSR and ladder BSE Euclidean wave function compared to those of the various QPEs. All solutions correspond to a bound state at $m_0 = 1.882m$.

For convenience, the FSR wave function is normalized according to the standard nonrelativistic one. The mass of the ground state for all cases is $m_0 = 1.882m$ and we take $\mu/m = 0.15$. At large separations we expect that the wave function behavior is essentially determined by the binding energy of the composite system. This is in agreement with the calculated results shown in Fig. 2. The main difference between the QPE predictions for short distances is due to the asymptotic behavior of their 2-particle free propagator $S_{\rm QPE}(q)$ for large values of q.

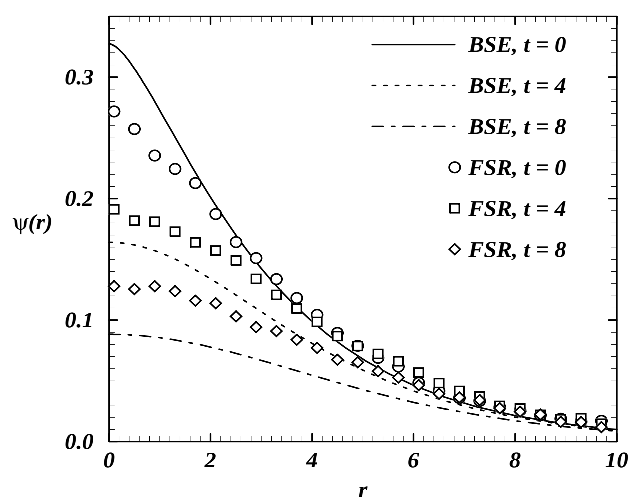

FIG. 3. FSR and ladder BSE Euclidean wave functions at three values of the Euclidean relative time t. The ground state mass for both calculations was at $m_0 = 1.882m$.

Effects of relativity in the dynamics are anticipated to play an important role in the relative time t dependence of the wave function, especially at small spatial separations between the constituents. In Fig. 3 we compare the ladder BSE and the FSR ground state wave functions for three values of the Euclidean relative time t. From this we see that the ladder BSE prediction falls off considerably faster in t as compared to the FSR result. This may be due to the fact that we need a substantially larger coupling constant in the BSE case to obtain the same binding energy. As a result the relativistic effects are enhanced in the interaction. At large distance both calculated wave functions agree essentially with each other and moreover show a very slow fall off in t, consistent with our expectation.

For actual hadronic systems the complication of fermions has also to be considered. Some progress has been achieved recently in including spin degrees of freedom within the FSR approach. It is clearly of great interest to study this further. In this paper we have presented for the scalar case the first calculations of bound state properties beyond the ladder approximation using the Feynman-Schwinger representation. When comparing our results to those of the Bethe-Salpeter equation in the ladder approximation, we find that the crossed-ladders significantly contribute to the binding energy.

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